Journal of Global Optimization (2005) 31:235–252

On Minimax Fractional Optimality and Duality with Generalized Convexity

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(Received 30 March 2004; accepted in revised form 1 April 2004)

Abstract. In this paper, we establish several sufficient optimality conditions for a class of generalized minimax fractional programming. Based on the sufficient conditions, a new dual model is constructed and duality results are derived. Our study naturally unifies and extends some previously known results in the framework of generalized convexity and dual models.

Mathematics Subject Classifications: 90C25, 90C32, 90C47.

Key words: Duality theorems, Generalized convexity, Generalized minimax fractional programming, Sufficient optimality conditions.

1. Introduction

We consider the following generalized fractional minimax problem

$$(\mathrm{P}) \qquad \text{minimize} \quad \sup_{y\in Y}\frac{f(x,y)}{h(x,y)} \text{subject to} \quad g(x) \leqslant 0,$$

where
(a) Y is a compact subset of $\mathbb{R}^m$,
(b) $f(\cdot,\cdot):\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is a differentiable function,
(c) $h(\cdot,\cdot):\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is a differentiable function,
(d) $g(\cdot):\mathbb{R}^n\to\mathbb{R}^p$ is a differentiable function.

In addition, we require

$$f(x,y)\geqslant 0 \quad \text{and} \quad h(x,y)>0 \quad \forall (x,y)\in X\times Y,$$

where $X=\{x\in\mathbb{R}^n : g(x)\leqslant 0\}$ is the set of feasible solutions of problem (P).

For the case of convex differentiabe minimax fractional programming, Yadav and Mukherjee (see Ref. [1]) formulated two dual models and established some duality results. Later, Chandra and Kumar (see Ref. [2]) pointed out certain omissions and inconsistencies in the formulation of Yadav and Mukherjee in [1], and they constructed two modified dual models and proved duality theorems for the convex differentiable fractional minimax programming. To relax the convexity assumptions in theorems on sufficient optimality conditions and duality, various generalized convexity concepts have been proposed. In [3] Liu, Wu and Sheu relaxed the convexity assumptions on the sufficient optimality in [2] and employed the optimality conditions to construct one parametric and two parametric-free dual models. They also established weak duality, strong duality, and strict converse duality theorems involving pseudoconvex and quasiconvex functions. Recently, Liu and Wu (see Refs. [4, 5]) derived the sufficient optimality conditions and duality theorems for the generalized fractional minimax problem in the framework of invex functions and (F, ρ)-convex functions, respectively. We also note that there were some new ideas of Bector, Suneja and Gupta [6], Rueda, Hanson and Singh [7] proposed on generalized convexity. In this paper,based on the ideas of Bector, Suneja and Gupta [6], Rueda, Hanson and Singh [7] and the concepts of (F, ρ, θ)- pseudoconvexity as well as (F, ρ, θ)-quasiconvexity in [5] we present the new concepts which extend the class of generalized convexity in [5–7]. Motivated by Chandra and Kumar [2] and Liu et al. [3–5], we introduce a new dual model for the generalized fractional minimax programming problem (P). Our dual model unifies two dual models of [4, 5] and includes some new dual models as special cases. We also establish sufficiency and duality for the generalized fractional minimax programming problem (P) with weakened convexity. Thus, this article unifies and extends the results of [4, 5] in the framework of generalized convexity and dual models.

The organization of the article is as follows. Some definitions and notations are given in Section 2. The sufficient optimality conditions are established in Section 3. By employing the sufficient conditions, we formulate a dual model and derive a number of duality results in Section 4.

2. Notations and Preliminary Results

Throughout the article we let

$$J = \{1, 2, \ldots, p\},$$

$$J(x) = \{j \in J : g_j(x) = 0\},$$

$$Y(x) = \left\{ y \in Y : \frac{f(x, y)}{h(x, y)} = \sup_{z \in Y} \frac{f(x, z)}{h(x, z)} \right\},$$

$$K = \{(s, t, \bar{y}) \in N \times \mathbb{R}^s_+ \times \mathbb{R}^{ms} : 1 \leqslant s \leqslant n + 1,$$

$$t = (t_1, \ldots, t_s) \in \mathbb{R}^s_+ \text{with } \sum_{i=1}^{s} t_i = 1 \text{ and}$$

$$\bar{y} = (y_1, \ldots, y_s) \text{ with } y_i \in Y(x),\ i = 1, \ldots, s \text{ for some } x \in X\}.$$

We assume the gradient $\nabla = \nabla_x$ is with respect to the variable x.

In order to establish the sufficient optimality conditions in Section 3, we first recall the following necessary conditions for optimality of (P) given by Chandra and Kumar in [2].

THEOREM 2.1 (Necessary conditions) [2]. *Let x^* be a* (P)-*optimal solution and let $\nabla g_j(x^*), j \in J(x^*)$ be linearly independent. Then there exist $(s^*; t^*; \bar{y}) \in K$, $v^* \in \mathbb{R}$ and $\mu^* \in \mathbb{R}^p_+$, such that*

$$\sum_{i=1}^{s^*} t_i^* \{\nabla f(x^*, y_i) - v^* \nabla h(x^*, y_i)\} + \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*) = 0, \tag{1}$$

$$f(x^*, y_i) - v^* h(x^*, y_i) = 0, \quad i = 1, \ldots, s^*, \tag{2}$$

$$\sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \tag{3}$$

$$\mu^* \in \mathbb{R}^p_+, \ \ t_i^* \geqslant 0, \ \ \sum_{i=1}^{s^*} t_i^* = 1, \ \ y_i \in Y(x^*), \quad i = 1, \ldots, s^*. \tag{4}$$

We also need the following definition in the sequel.

DEFINITION 2.1. A functional $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ (where $X \subseteq \mathbb{R}^n$) is said to be sublinear if for $(x, x_0) \in X \times X$,

$$F(x, x_0; a_1 + a_2) \leqslant F(x, x_0; a_1) + F(x, x_0; a_2) \quad \forall a_1, a_2 \in \mathbb{R}^n$$

and

$$F(x, x_0; \alpha a) = \alpha F(x, x_0; a) \quad \forall \alpha \in \mathbb{R}, \ \ \alpha \geqslant 0 \text{ and } \ a \in \mathbb{R}^n.$$

It is obvious that $F(x, x_0; 0) = 0$.

3. Sufficient Conditions

In this section, we obtain sufficient optimality conditions for (P) based on the ideas of Rueda, Hanson and Singh in [7].

Let $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ be sublinear, $\phi_0, \phi_1 : \mathbb{R} \to \mathbb{R}$, $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $b_0, b_1 : X \times X \to \mathbb{R}_+$. Let ρ_0, ρ_1 be real numbers.

THEOREM 3.1 (Sufficient condition). *Suppose that $(x^*, v^*, \mu^*, s^*, t^*, \bar{y})$ satisfies relations (1)–(4) and there exist $F, \theta, \phi_0, b_0, \rho_0$* and *$\phi_1, b_1, \rho_1$ such that*

$$F\left(x, x^*; \sum_{i=1}^{s^*} t_i^* \nabla(f(x^*, y_i) - v^* h(x^*, y_i))\right) \geqslant -\rho_0 \|\theta(x, x^*)\|^2$$
$$\implies b_0(x, x^*)\phi_0\left(\sum_{i=1}^{s^*} t_i^*(f(x, y_i) - v^* h(x, y_i)) - \sum_{i=1}^{s^*} t_i^*(f(x^*, y_i) - v^* h(x^*, y_i))\right) \geqslant 0 \tag{5}$$

and

$$-b_1(x, x^*)\phi_1\left(\sum_{j=1}^{p} \mu_j^* g_j(x^*)\right) \leqslant 0 \implies F\left(x, x^*; \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right) \leqslant -\rho_1 \|\theta(x, x^*)\|^2. \tag{6}$$

Further, assume

$$a \geqslant 0 \implies \phi_1(a) \geqslant 0, \tag{7}$$
$$\phi_0(a) \geqslant 0 \implies a \geqslant 0, \tag{8}$$
$$b_0(x, x^*) \geq 0, \quad b_1(x, x^*) > 0, \tag{9}$$
$$\rho_0 + \rho_1 \geqslant 0. \tag{10}$$

Then x^* is an optimal solution of (P).

Proof. We proceed by contradiction. Suppose to the contrary that x^* is not an optimal solution for (P). Then there exists a (P)-feasible point x, such that

$$v^* = \frac{f(x^*; y_i)}{h(x^*; y_i)} > \sup_{y \in Y} \frac{f(x, y)}{h(x, y)}, \quad i = 1, \ldots, s^*.$$

Thus, we have

$$f(x, y) - v^* h(x, y) < 0 \;\; \forall y \in Y. \tag{11}$$

By (2), (4) and (11), we obtain

$$\sum_{i=1}^{s^*} t_i^*(f(x, y_i) - v^* h(x, y_i)) < \sum_{i=1}^{s^*} t_i^*(f(x^*, y_i) - v^* h(x^*, y_i)). \tag{12}$$

On the other hand, from (3), (7) and (9) we have

$$-b_1(x,x^*)\phi_1\left(\sum_{j=1}^{p}\mu_j^* g_j(x^*)\right) \leqslant 0,$$

which by (6) implies

$$F\left(x,x^*;\sum_{j=1}^{p}\mu_j^*\nabla g_j(x^*)\right) \leqslant -\rho_1\|\theta(x,x^*)\|^2.$$

It follows from (1), the sublinearity of F and (10) that

$$F\left(x,x^*;\sum_{i=1}^{s^*}t_i^*(\nabla f(x^*,y_i) - v^*\nabla h(x^*,y_i))\right) \geqslant -\rho_0\|\theta(x,x^*)\|^2.$$

Then, by (5) we have

$$b_0(x,x^*)\phi_0\left(\sum_{i=1}^{s^*}t_i^*(f(x,y_i)-v^*h(x,y_i))-\sum_{i=1}^{s^*}t_i^*(f(x^*,y_i)-v^*h(x^*,y_i))\right) \geqslant 0.$$

Finally from (8) and (9), we conclude that

$$\sum_{i=1}^{s^*}t_i^*(f(x,y_i)-v^*h(x,y_i))-\sum_{i=1}^{s^*}t_i^*(f(x^*,y_i)-v^*h(x^*,y_i)) \geqslant 0,$$

which is a contradiction to (12). Therefore, x^* is an optimal solution for (P). The theorem is proved.

Remark 3.1. If $\phi_0(t)=\phi_1(t)=t$ and $b_0(x,x^*)=b_1(x,x^*)=1$, then clearly Theorem 3.1 reduce to Theorem 3.1 in [5]. If $\phi_0(t)=\phi_1(t) = t,\ b_0(x,x^*)=b_1(x,x^*)=1, \rho_0=\rho_1=0, F(x,u;a)=\eta_(x,u)^T a$, where η is a functional from $X\times X$ to $\mathbb{R}^n$, then Theorem 3.1 reduce to Theorem 2.2 in [4].

The following example shows that the conditions of Theorem 3.1 are weaker than the sufficient conditions in [4, 5].

EXAMPLE 3.1. Let $f(x,y)=\cos x+y^2+1, h(x,y)=1, g_i(x)=0, i=1,2,\ldots,p, Y=[-1,1], X=(0,2\pi)$. It is obvious that $Y(x)=\{-1,1\}$. Let $\psi(x)=\sup_Y f(x,y)/h(x,y)$. Then $\psi(x)=\cos x+2$. We note that $x=\pi$ is an optimal solution for $\min_X\sup_Y f(x,y)/h(x,y)$. If $F(x,x^*,y)=\eta(x,x^*)^T y$, and $\eta(x,y)=y-x, \rho_0=0, \rho_1=0$, and $\theta=0$, then $\psi(x)$ is not (F,ρ_0,θ)-pseudoconvex at $x^*=\pi$. Thus, using the result in [4, 5], we can not show that $x^*=\pi$ is an optimal solution of $\min_X\sup_Y f(x,y)/h(x,y)$. However, if we define

$$b_0(x,x^*)=\begin{cases}-1, & x^*\leqslant x\leqslant\pi,\ 0<x^*<\pi,\\ 0, & x<\pi,\ 0<x^*<\pi,\\ -1, & \pi<x\leqslant x^*,\ \pi<x^*<2\pi,\\ 0, & x\leqslant\pi,\ \pi<x^*<2\pi,\end{cases}$$

$\phi_0(t)=\phi_1(t)=t, b_1(x,x^*)=1, \rho_0=\rho_1=0$, then, $f(x,y)$ and $h(x,y)=1$ satisfy the conditions of Theorem 3.1 at $x^*=\pi$. Therefore $x^*=\pi$ is an optimal solution for $\min_X\sup_Y f(x,y)$.

THEOREM 3.2. *Suppose that* $(x^*,v^*,\mu^*,s^*,t^*,\bar{y})$ *satisfy relations* (1)–(4) *and there exist* $F,\theta,\phi_0,b_0,\rho_0$ *and* ϕ_1,b_1,ρ_1 *such that*

$$F\left(x,x^*;\sum_{i=1}^{s^*}t_i^*\nabla(f(x^*,y_i)-v^*h(x^*,y_i))\right)\geqslant-\rho_0\|\theta(x,x^*)\|^2$$
$$\Longrightarrow b_0(x,x^*)\phi_0\left(\sum_{i=1}^{s^*}t_i^*(f(x,y_i)-v_h(x,y_i))\right.$$
$$\left.-\sum_{i=1}^{s^*}t_i^*(f(x^*,y_i)-v^*h(x^*,y_i))\right)>0, \quad (13)$$

or equivalently,

$$b_0(x,x^*)\phi_0\left(\sum_{i=1}^{s^*}t_i^*(f(x,y_i)-v^*h(x,y_i))-\sum_{i=1}^{s^*}t_i^*(f(x^*,y_i)-v^*h(x^*,y_i))\right)\leqslant0$$
$$\Longrightarrow F\left(x,x^*;\sum_{i=1}^{s^*}t_i^*\nabla(f(x^*,y_i)-v^*h(x^*,y_i))\right)<-\rho_0\|\theta(x,x^*)\|^2$$
$$(14)$$

and

$$-b_1(x,x^*)\phi_1\left(\sum_{j=1}^{p}\mu_j^*g_j(x^*)\right)\leqslant0\ \Longrightarrow\ F\left(x,x^*;\sum_{j=1}^{p}\mu_j^*\nabla g_j(x^*)\right)$$
$$\leqslant-\rho_1\|\theta(x,x^*)\|^2. \quad (15)$$

Further, assume (7), (9), (10) *and*

$$a\leqslant0\ \Longrightarrow\ \phi_0(a)\leqslant0, \quad (16)$$

are satisfied. Then x^* *is an optimal solution of* (P).

Proof. Suppose that x^* is not an optimal solution for (P). Then there exists a (P)-feasible point x, such that

$$v^*=\frac{f(x^*,y_i)}{h(x^*,y_i)}>\sup_{y\in Y}\frac{f(x,y)}{h(x,y)},\quad i=1,\dots,s^*.$$

Thus, we have

$$f(x,y) - v^*h(x,y) < 0 \qquad \forall y \in Y. \tag{17}$$

By (2), (4) and (17), we obtain

$$\sum_{i=1}^{s^*} t_i^*(f(x,y_i) - v^*h(x,y_i)) < \sum_{i=1}^{s^*} t_i^*(f(x^*,y_i) - v^*h(x^*,y_i)). \tag{18}$$

Using (18), (16), (9) and (14), we have

$$F\left(x,x^*;\sum_{i=1}^{s^*} t_i^*\nabla(f(x^*,y_i) - v^*h(x^*,y_i))\right) < -\rho_0\|\theta(x,x^*)\|^2. \tag{19}$$

By (9), (7), (15) and (3) it follows that

$$F\left(x,x^*;\sum_{j=1}^{p} \mu_j^*\nabla g_j(x^*)\right) \leqslant -\rho_1\|\theta(x,x^*)\|^2. \tag{20}$$

Now, on adding (19) and (20), and making use of the sublinearity of F and (10), we obtain

$$F\left(x,x^*;\sum_{i=1}^{s^*} t_i^*\nabla(f(x^*,y_i) - v^*h(x^*,y_i)) + \sum_{j=1}^{p} \mu_j^*\nabla g_j(x^*)\right) < 0.$$

On the other hand, (1) implies

$$F(x,x^*;\sum_{i=1}^{s^*} t_i^*\nabla(f(x^*,y_i) - v^*h(x^*,y_i) + \sum_{j=1}^{p} \mu_j^*\nabla g_j(x^*)) = 0.$$

Hence we have a contradiction. Therefore, x^* is an optimal solution for (P). This completes the proof of the theroem.

THEOREM 3.3. *Suppose that* $(x^*,v^*,\mu^*,s^*,t^*,\bar{y})$ *satisfy relations* (1)–(4) *and there exist* $F,\theta,\phi_0,b_0,\rho_0$ *and* ϕ_1,b_1,ρ_1 *such that*

$$b_0(x,x^*)\phi_0\left(\sum_{i=1}^{s^*} t_i^*(f(x,y_i)-v^*h(x,y_i))-\sum_{i=1}^{s^*} t_i^*(f(x^*,y_i)-v^*h(x^*,y_i))\right)\leqslant 0$$

$$\Longrightarrow F\left(x,x^*;\sum_{i=1}^{s^*} t_i^*\nabla(f(x^*,y_i)-v^*h(x^*,y_i))\right)\leqslant -\rho_0\|\theta(x,x^*)\|^2 \tag{21}$$

and

$$F\left(x,x^*;\sum_{j=1}^{p} \mu_j^*\nabla g_j(x^*)\right) \geqslant -\rho_1\|\theta(x,x^*)\|^2$$

$$\Longrightarrow -b_1(x,x^*)\phi_1\left(\sum_{j=1}^{p} \mu_j^* g_j(x^*)\right) > 0. \tag{22}$$

Further, assume (7), (9), (10) *and* (16) *are satisfied. Then* x^* *is an optimal solution of* (P).

Proof. The proof is similar to that of Theorem 3.2 and hence omitted.

4. Duality Theorems

In this section, we present a new dual model for (P) and establish weak, strong, and strict converse duality results for (P).

To unify and extend the dual models in [4, 5], we need to divide $\{1, 2, \ldots, p\}$ into several parts. Let $J_\alpha (0 \leqslant \alpha \leqslant r)$ be a partition of $\{1, 2, \ldots, p\}$, that is,

$$J_\alpha \neq J_\beta \text{ for } \alpha \neq \beta \quad \text{and} \quad \bigcup_{\alpha=0}^{r} J_\alpha = \{1, 2, \ldots, p\}.$$

We note that for (P)-optimal x^*, (3) implies

$$\sum_{j \in J_\alpha} \mu_j^* g_j(x^*) = 0, \quad \alpha = 0, 1, \ldots, r.$$

We now recast the necessary conditions in Theorem 2.1 in the following form.

LEMMA 4.1 (Necessary conditions). *Let* x^* *be a* (P)-*optimal solution and let* $\nabla g_j(x^*), j \in J(x^*)$ *be linearly independent. Then there exist* $(s^*, t^*, \bar{y}) \in K, v^* \in \mathbb{R}$, and $\mu^* \in \mathbb{R}_+^p$ *such that*

$$\left(\sum_{i=1}^{s^*} t_i^* h(x^*, y_i)\right) \nabla \left(\sum_{i=1}^{s^*} t_i^* f(x^*, y_i) + \sum_{j=1}^{p} \mu_j^* g_j(x^*)\right) - \left(\sum_{i=1}^{s^*} t_i^* f(x^*, y_i) + \sum_{j \in J_0} \mu_j^* g_j(x^*)\right) \nabla \left(\sum_{i=1}^{s^*} t_i^* h(x^*, y_i)\right) = 0, \quad (23)$$

$$\sum_{j \in J_\alpha} \mu_j^* g_j(x^*) = 0, \quad \alpha = 1, \ldots, r, \quad (24)$$

$$\mu^* \in \mathbb{R}_+^p, \ \ t_i^* \geqslant 0, \ \ \sum_{i=1}^{s^*} t_i^* = 1, \ \ y_i \in Y(x^*), \quad i = 1, \ldots, s^*, \quad (25)$$

where $J_\alpha (0 \leqslant \alpha \leqslant r)$ *is a partition of* $\{1, 2, \ldots, p\}$.

Proof. It suffices to establish (23). From (1) and (2), we get

$$\nabla \sum_{i=1}^{s^*} t_i^* f(x^*, y_i) - \frac{f(x^*, y_k)}{h(x^*, y_k)} \nabla \sum_{i=1}^{s^*} t_i^* h(x^*, y_i) + \nabla \sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \quad k = 1, \ldots, s^*.$$

Multiplying the respective equation above by $t_i h(x^*, y_i), i = 1, \ldots, s^*$, and adding them altogether, we have

$$\left(\sum_{i=1}^{s^*} t_i^* h(x^*, y_i) \right) \nabla \left(\sum_{i=1}^{s^*} t_i^* f(x^*, y_i) + \sum_{j=1}^{p} \mu_j^* g_j(x^*) \right) - \left(\sum_{i=1}^{s^*} t_i^* f(x^*, y_i) \right) \nabla \left(\sum_{i=1}^{s^*} t_i^* h(x^*, y_i) \right) = 0.$$

The above equation together with (3) implies

$$\left(\sum_{i=1}^{s^*} t_i^* h(x^*, y_i) \right) \nabla \left(\sum_{i=1}^{s^*} t_i^* f(x^*, y_i) + \sum_{j=1}^{p} \mu_j^* g_j(x^*) \right) - \left(\sum_{i=1}^{s^*} t_i^* f(x^*, y_i) + \sum_{j \in J_0} \mu_j^* g_j(x^*) \right) \nabla \left(\sum_{i=1}^{s^*} t_i^* h(x^*, y_i) \right) = 0.$$

The theorem is proved.

Our dual model is as follows.

$$\text{(D)} \qquad \max_{(s,t,\bar{y}) \in K} \sup_{(z,\mu) \in H(s,t,\bar{y})} \frac{\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j \in J_0} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i h(z, y_i)},$$

where $H(s, t, \bar{y})$ denotes the set of all $(z, \mu) \in \mathbb{R}^n \times \mathbb{R}_+^p$ satisfying

$$\left(\sum_{i=1}^{s} t_i h(z, y_i) \right) \nabla \left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z) \right) - \left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j \in J_0} \mu_j g_j(z) \right) \nabla \left(\sum_{i=1}^{s} t_i h(z, y_i) \right) = 0, \tag{26}$$

$$\sum_{j \in J_\alpha} \mu_j g_j(z) \geqslant 0, \quad \alpha = 1, \ldots, r, \tag{27}$$

$$J_\alpha \neq J_\beta \text{ for } \alpha \neq \beta \text{ and } \bigcup_{\alpha=0}^{r} J_\alpha = \{1, 2, \ldots, p\}.$$

THEOREM 4.1 (Weak duality). *Let* x *and* $(z, \mu, s, t, \bar{y})$ *be* (P)*-feasible and be* (D)*-feasible, respectively. Suppose there exist* $F, \theta, \phi_0, b_0, \rho_0$ *and* $\phi_\alpha, b_\alpha, \rho_\alpha, \alpha = 1, \ldots, r$, *such that*

$$F\Bigg(x, z; (\sum_{i=1}^{s} t_i h(z, y_i))\nabla(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j\in J_0} \mu_j g_j(z))$$
$$\left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j\in J_0} \mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i h(z, y_i))\right) \geqslant -\rho_0\|\theta(x, z)\|^2$$
$$\Longrightarrow\ b_0(x, z)\phi_0\left(\left(\sum_{i=1}^{s} t_i h(z, y_i\right)\right)\left(\sum_{i=1}^{s} t_i f(x, y_i) + \sum_{j\in J_0} \mu_j g_j(x)\right).$$
$$-\left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j\in J_0} \mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i h(x, y_i)\right)\Bigg) \geqslant 0 \tag{28}$$

and

$$-b_\alpha(x, z)\phi_\alpha\left(\left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\left(\sum_{j\in J_\alpha} \mu_j g_j(z)\right)\right) \leqslant 0$$
$$\Longrightarrow\ F\left(x, z; \left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\left(\sum_{j\in J_\alpha} \mu_j \nabla g_j(z)\right)\right)$$
$$\leqslant -\rho_\alpha\|\theta(x, z)\|^2, \quad \alpha = 1, \ldots, r. \tag{29}$$

Further, assume

$$a \geqslant 0 \Longrightarrow \phi_\alpha(a) \geqslant 0, \quad \alpha = 1, \ldots, r, \tag{30}$$

$$\phi_0(a) \geqslant 0 \ \Longrightarrow\ a \geqslant 0, \tag{31}$$

$$b_0(x, z) > 0, \qquad b_\alpha(x, z) \geqslant 0, \quad \alpha = 1, \ldots, r, \tag{32}$$

$$\rho_0 + \sum_{\alpha=1}^{r} \rho_\alpha \geqslant 0. \tag{33}$$

Then

$$\sup_{y\in Y} \frac{f(x, y)}{h(x, y)} \geqslant \frac{\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j\in J_0} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i h(z, y_i)}.$$

Proof. Suppose to the contrary that

$$\sup_{y\in Y} \frac{f(x, y)}{h(x, y)} < \frac{\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j\in J_0} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i h(z, y_i)}.$$

Thus we have an inequality

$$\left(\sum_{i=1}^{s} t_i h(z, y_i)\right) f(x, y) < \left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j \in J_0} \mu_j g_j(z)\right) h(x, y) \quad \forall y \in Y.$$

Furthermore, this implies

$$\left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\left(\sum_{i=1}^{s} t_i f(x, y_i)\right) < \left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j \in J_0} \mu_j g_j(z)\right) \times \left(\sum_{i=1}^{s} t_i h(x, y_i)\right).$$

Hence, we have

$$\left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\left(\sum_{i=1}^{s} t_i f(x, y_i) + \sum_{j \in J_0} \mu_j g_j(x)\right) - \left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j \in J_0} \mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i h(x, y_i)\right) < \left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\left(\sum_{j \in J_0} \mu_j g_j(x)\right).$$

Using the fact that $\sum_{i=1}^{s} t_i h(z, y_i) > 0, \sum_{j \in J_0} \mu_j g_j(x) \leqslant 0,$ and the last inequality, we have

$$\left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\left(\sum_{i=1}^{s} t_i f(x, y_i) + \sum_{j \in J_0} \mu_j g_j(x)\right) - \left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j \in J_0} \mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i h(x, y_i)\right) < 0. \tag{34}$$

From (34), (31), (32) and (28), we get

$$F\left(x, z; \left(\sum_{i=1}^{s} t_i h(z, y_i)\right) \nabla\left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j \in J_0} \mu_j g_j(z)\right) - \left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j \in J_0} \mu_j g_j(z)\right) \nabla\left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\right) < -\rho_0 \|\theta(x, z)\|^2. \tag{35}$$

Using $\sum_{i=1}^{s} t_i h(z, y_i) > 0$, (27), (32) and (30) we get

$$-b_\alpha(x,z)\phi_\alpha\left(\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\left(\sum_{j\in J_\alpha}\mu_j g_j(z)\right)\right) \leqslant 0, \quad \alpha = 1,\ldots,r,$$

and whence from (29), we have

$$F\left(x,z;\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\left(\sum_{j\in J_\alpha}\mu_j \nabla g_j(z)\right)\right) \leqslant -\rho_\alpha\|\theta(x,z)\|^2, \quad \alpha = 1,\ldots,r. \tag{36}$$

Now, on adding (35) and (36), and utilizing the sublinearity of F and (33), we obtain

$$F\left(x,z;\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\nabla\left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j=1}^{p}\mu_j g_j(z)\right)\right.$$
$$\left.-\left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\right) < 0,$$

which is a contradiction to (26). This completes the proof.

Similar to the proof of Theorem 4.1, we can establish Theorem 4.2. Therefore, we simply state it here without proof.

THEOREM 4.2 (Weak duality). *Let* x *and* $(z,\mu,s,t,\bar{y})$ *be* (P)-*feasible and be* (D)-*feasible, respectively. Suppose there exist* $F,\theta,\phi_0,b_0,\rho_0$ *and* $\phi_\alpha,b_\alpha,\rho_\alpha,\alpha = 1,\ldots,r$, *such that*

$$b_0(x,z)\phi_0\left(\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\left(\sum_{i=1}^{s} t_i f(x,y_i) + \sum_{j\in J_0}\mu_j g_j(x)\right)\right.$$
$$\left.-\left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i h(x,y_i)\right)\right) < 0$$
$$\Longrightarrow F\left(x,z;\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\nabla\left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)\right.$$
$$\left.-\left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\right) \leqslant -\rho_0\|\theta(x,z)\|^2$$

and

$$- b_{\alpha}(x,z)\phi_{\alpha}\left((\sum_{j\in J_{\alpha}} \mu_j g_j(z))(\sum_{i=1}^{s} t_i h(z,y_i)) \right) \leqslant 0$$

$$\Longrightarrow \quad F\left(x,z;(\sum_{i=1}^{s} t_i h(z,y_i))(\sum_{j\in J_{\alpha}} \mu_j \nabla g_j(z)) \right)$$

$$< -\rho_{\alpha}\|\theta(x,z)\|^2, \quad \alpha = 1,\ldots,r.$$

Further, assume (ref34), (31), (32) *and* (33) *are satisfied. Then*

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)} \geqslant \frac{\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)}{\sum_{i=1}^{s} t_i h(z,y_i)}.$$

THEOREM 4.3 (Strong duality). *Assume that* x^* *is a* (P)-*optimal solution and* $\nabla g_j(x^*), j \in J(x^*)$ *is linearly independent. Then there exist* $(s^*, t^*, \bar{y}) \in K, (x^*, \mu^*) \in H(s^*, t^*, \bar{y})$ *such that* $(x^*, \mu^*, t^*, \bar{y})$ *is a* (D)-*optimal solution. If, in addition, the hypothesis of Theorem* 4.1 *or Theorem* 4.2 *holds for a* (D)-*feasible point* $(z, \mu, v, s, t, \bar{y})$, *then the two problems* (P) *and* (D) *have the same extremal values.*

Proof. By Lemma 4.1, there exist $(s^*, t^*, \bar{y}) \in K, (x^*, \mu^*) \in H(s^*, t^*, \bar{y})$ such that $(x^*, \mu^*, s^*, t^*, \bar{y})$ is a feasible solution for (D), optimality of this feasible solution for (D) follows from Theorem 4.1 or Theorem 4.2 accordingly.

THEOREM 4.4 (Strict converse duality). *Let* x^* *and* $(z, \mu, s, t, \bar{y})$ *be optimal solutions of* (P) *and* (D), *respectively. Suppose that* $\nabla g_j(x^*), j \in J(x^*)$ *is linearly independent, and there exist* $F, \theta, \phi_0, b_0, \rho_0$ *and* $\phi_{\alpha}, b_{\alpha}, \rho_{\alpha}, \alpha = 1,\ldots,r$, *such that*

$$F\left(x^*,z;\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\nabla\left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)\right.$$

$$\left. -\left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\right) \geqslant -\rho_0\|\theta(x^*,z)\|^2$$

$$\Longrightarrow \quad b_0(x^*,z)\phi_0\left(\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\left(\sum_{i=1}^{s} t_i f(x^*,y_i) + \sum_{j\in J_0}\mu_j g_j(x^*)\right)\right.$$

$$\left. -\left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i h(x^*,y_i)\right)\right) \geqslant 0 \tag{37}$$

and

$$-\, b_\alpha(x^*,z)\phi_\alpha\left(\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)(\sum_{j\in J\alpha}\mu_j g_j(z))\right) \leqslant 0$$

$$\Longrightarrow \quad F\left(x^*,z;\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\left(\sum_{j\in J\alpha}\mu_j \nabla g_j(z)\right)\right)$$

$$\leqslant -\rho_\alpha\|\theta(x^*,z)\|^2, \quad \alpha = 1,\ldots,r. \tag{38}$$

Further, assume (30), (32), (33) *and*

$$\phi_0(a) \geqslant 0 \implies a > 0. \tag{39}$$

Then $x^* = z$; that is, z is a (P)-optimal solution.

Proof. We shall assume that $x^* \neq z$ and reach a contradiction. From Theorem 4.3, we know that

$$\sup_{y\in Y}\frac{f(x^*,y)}{h(x^*,y)} = \frac{\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)}{\sum_{i=1}^{s} t_i h(z,y_i)}.$$

Thus we have

$$\left(\sum_{i=1}^{s} t_i h(z,y_i)\right) f(x^*,y) \leqslant \left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)\right) h(x^*,y) \quad \forall y \in Y.$$

This further implies

$$\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\left(\sum_{i=1}^{s} t_i f(x^*,y_i)\right) \leqslant \left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)$$
$$\times \left(\sum_{i=1}^{s} t_i h(x^*,y_i)\right).$$

Hence, we have

$$\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\left(\sum_{i=1}^{s} t_i f(x^*,y_i) + \sum_{j\in J_0}\mu_j g_j(x^*)\right)$$
$$-\left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i h(x^*,y_i)\right)$$
$$\leqslant \left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\left(\sum_{j\in J_0}\mu_j g_j(x^*)\right).$$

Using the fact that $\sum_{i=1}^{s} t_i h(z,y_i) > 0, \sum_{j\in J_0}\mu_j g_j(x^*) \leqslant 0$ and the last inequality, we have

$$\left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\left(\sum_{i=1}^{s} t_i f(x^*, y_i) + \sum_{j\in J_0} \mu_j g_j(x^*)\right) - \left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j\in J_0} \mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i h(x^*, y_i)\right) \leqslant 0. \quad (40)$$

From (40), (39), (32) and (37), we get

$$F\left(x^*, z; \left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\nabla\left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0} \mu_j g_j(z)\right)\right) - \left(\sum_{i=1}^{s} t_i f(z,y_i) + \sum_{j\in J_0} \mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i h(z,y_i)\right) < -\rho_0\|\theta(x^*, z)\|^2. \quad (41)$$

Using $\sum_{i=1}^{s} t_i h(z, y_i) > 0$, (27), (30) and (32) we get

$$-b_\alpha(x^*, z)\phi_\alpha\left(\left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\left(\sum_{j\in J_\alpha} \mu_j g_j(z)\right)\right) \leqslant 0, \quad \alpha = 1, \dots, r.$$

From (38), we have

$$F\left(x^*, z; \left(\sum_{i=1}^{s} t_i h(z, y_i)\right) \sum_{j\in J_\alpha} \mu_j \nabla g_j(z)\right) \leqslant -\rho_\alpha\|\theta(x^*, z)\|^2, \quad \alpha = 1, \dots, r. \quad (42)$$

Now, on adding (41) and (42), and utilizing the sublinearity of F and (33), we obtain

$$F\left(x^*, z; \left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\nabla\left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z)\right) - \left(\sum_{i=1}^{s} t_i f(z, y_i) + \sum_{j\in J_0} \mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i h(z, y_i)\right)\right) < 0,$$

which is a contradiction to (30). The proof is completed.

Similar to the proof of Theorem 4.4, we can establish Theorem 4.5. Therefore, we simply state it here without proof.

THEOREM 4.5 (Strict converse duality). *Let* x^* *and* $(z, \mu, s, t, \vec{y})$ *be optimal solutions of* (P) *and* (D), *respectively, and suppose that* $\nabla g_j(x^*), j \in J(x^*)$ *is linearly independent, and there exist* $F, \theta, \phi_0, b_0, \rho_0$ and $\phi_\alpha, b_\alpha, \rho_\alpha, \alpha = 1, \dots, r$, such that

$$b_0(x^*,z)\phi_0\left(\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\left(\sum_{i=1}^{s} t_i f(x^*,y_i)+\sum_{j\in J_0}\mu_j g_j(x^*)\right)\right.$$
$$\left.-\left(\sum_{i=1}^{s} t_i f(z,y_i)+\sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i h(x^*,y_i)\right)\right)<0$$
$$\Longrightarrow F\left(x^*,z;\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\nabla\left(\sum_{i=1}^{s} t_i f(z,y_i)+\sum_{j\in J_0}\mu_j g_j(z)\right)\right.$$
$$\left.-\left(\sum_{i=1}^{s} t_i f(z,y_i)+\sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\right)\leqslant-\rho_0\|\theta(x^*,z)\|^2$$

and

$$-b_\alpha(x^*,z)\phi_\alpha\left(\left(\sum_{i=1}^{s} t_i h(z,y_i))(\sum_{j\in J_\alpha}\mu_j g_j(z)\right)\right)\leqslant 0$$
$$\Longrightarrow F\left(x^*,z;\left(\sum_{i=1}^{s} t_i h(z,y_i)\right)\left(\sum_{j\in J_\alpha}\mu_j\nabla g_j(z)\right)\right)<-\rho_\alpha\|\theta(x^*,z)\|^2,\ \alpha=1,\ldots,r.$$

Further, assume

$$a\geqslant 0 \implies \phi_\alpha(a)\geqslant 0,\alpha=1,\ldots,r$$
$$\phi_0(a)\geqslant 0 \implies a>0,$$
$$b_0(x^*,z)>0,\ b_\alpha(x^*,z)\geqslant 0,\quad \alpha=1,\ldots,r,$$
$$\rho_0+\sum_{\alpha=1}^{r}\rho_\alpha\geqslant 0.$$

Then $x^*=z$; that is, z is a (P)*-optimal solution.*

Remark 4.1. (i) If $\phi_0(t)=\phi_\alpha(t)=t,\alpha=1,\ldots,r; b_0(x,z)=b_\alpha(x,z)=1$, $\alpha=1,\ldots,r; J_0=\emptyset, J_\beta=\{1,\ldots,p\}$ for some $\beta, J_\alpha=\emptyset$ for $\alpha\neq\beta; F(x,z;a)=\eta(x,z)^T a$; and $\rho_\alpha=0,\alpha=0,1,\ldots,r$, then it is obvious that theorem 4.1–4.5 above reduce to Theorem 3.1–3.3 in [4]. If $\phi_0(t)=\phi_\alpha(t)=t$, $\alpha=1,\ldots,r; b_0(x,z)=b_\alpha(x,z)=1,\alpha=1,\ldots,r; J_0=\emptyset,\ J_\beta=\{1,\ldots,p\}$ for some $\beta, J_\alpha=\emptyset$ for $\alpha\neq\beta$; then it is obvious that theorem 4.1–4.5 above reduce to Theorem 4.1–4.3 in [5].

(ii) If $\phi_0(t)=\phi_\alpha(t)=t,\alpha=1,\ldots,r; b_0(x,z)=b_\alpha(x,z)=1,\ \alpha=1,\ldots,r$; $J_0=\{1,\ldots,p\}, J_\alpha=\emptyset,\alpha=1,\ldots,r; F(x,z;a)=\eta(x,z)^T a$, and $\rho_\alpha=0,\alpha=0,1,\ldots,r$, then it is obvious that theorem 4.1–4.5 above reduce to Theorem 5.2–5.4 in [4]. If $\phi_0(t)=\phi_\alpha(t)=t,\alpha=1,\ldots,r; b_0(x,z)=b_\alpha(x,z)=1$, $\alpha=1,\ldots,r; J_0=\{1,\ldots,p\}, J_\alpha=\emptyset\ \alpha=0,1,\ldots,r$, then it is obvious that theorem 4.1–4.5 above reduce to Theorem 4.9–4.11 in [5].

From Remarks 3.1 and 4.1, we conclude that the results obtained in this article unifies and extends those of [4, 5] in the framework of generalized convexity and dual models.

5. Discussion

In this paper, we present some new classes of generalized convexity and a unified dual model for the generalized fractional minimax programming problem. Based on our formulation, we also obtain sufficient optimality conditions and derive a number of duality results for the generalized fractional minimax programming problem under the assumptions of generalized convexity. It is noted that the previously known results in [4, 5] are now some special cases of our results. In fact, by appropriate choices of the partitioning sets $J_0, J_\alpha, \alpha = 1, 2, \ldots, r$, we are able to obtain a number of interesting new situations.

We remark that Lai and Lee [9] and Lai et al. [10] had considered general non differentiable minimax fractional programming problem in the form:

$$(P)' \qquad \text{minimize} \quad \sup_{y \in Y} \frac{f(x,y) + (x^T A x)^{\frac{1}{2}}}{h(x,y) - (x^T B x)^{\frac{1}{2}}}$$
$$\text{subject to} \quad g(x) \leqslant 0,$$

where A and B are $n \times n$ positive semidefinite matrices satisfying

$$f(x,y) + (x^T A x)^{1/2} \geqslant 0 \quad \text{and} \quad h(x,y) - (x^T B x)^{1/2} > 0$$
$$\forall (x,y) \in X = \{x \in \mathbb{R}^n : g(x) \leqslant 0\}.$$

They introduced two dual models for problem (P)' and proved duality theorems under convexity, pseudoconvexity and quasiconvexity conditions. It will be interesting to see whether or not the dual models and the corresponding results developed in this paper still hold for the above nondifferentiable minimax fractional programming problem (P)'.

We also note that the minimax fractional programming problem (P) in this paper reduce to the generalized fractional programming problem (P) in [11, 12] if Y is a finite index set. And our results greatly improve, unify and extend the work in [11]. Meanwhile, based on the ideas in [12], the study on saddle-point type optimality criteria for minimax fractional programming problem (P) will be pursued in future research.

6. Acknowledgements

This research was partially supported by the National Natural Science Foundation of China, NCET of Ministry of Education P.R.China and the

Natural Foundation of Chongqing. The authors are grateful to referees for their constructive comments helping to give the present form to the paper.

References

1. Yadav, S.R. and Mukherjee, R.N. (1990), Duality for fractional minimax programming problems, *Journal of Australian Mathematical Society, Series B*, 31, 484–492.
2. Chandra, S. and Kumar, V. (1995), Duality in fractional minimax programming, *Journal of Australian Mathematical Society, Series A*, 58, 376–386.
3. Liu, J.C., Wu, C.S. and Sheu, R.L. (1997), Duality for fractional minimax programming, *Optimization*, 41, 117–133.
4. Liu, J.C. and Wu, C.S. (1998), On minimax fractional optimality conditions with invexity, *Journal of Mathematical Analysis and Applications*, 219, 21–35.
5. Liu, J.C. and Wu, C.S. (1998), On minimax fractional optimality conditions with (F, ρ) convexity, *Journal of Mathematical Analysis and Applications*, 219, 36–51.
6. Bector, C.R. Chandra, S., Suneja, S.K. and Gupta, S. (1992), Univex sets, functions and univex nonlinear programming, *Proceedings of the Administrative Sciences Association of Canada*, 115–124.
7. Rueda, N.G., Hanson, N.G. and Singh, C. (1995), Optimality and duality with generalized convexity, *Journal of Optimization Theory and Applications*, 86, 491–500.
8. Hanson, M.A. and Mond, B. (1987), Necessary and sufficient conditions in constained optimization, *Mathematical Programming*, 37, 51–58.
9. Lai, H.C., Liu, J.C. and Tanaka, K. (1999), Necessary and sufficent conditions for minimax fractional programming, *Journal of Mathematical Analysis and Applications*, 230, 311–328.
10. Lai, H.C. and Lee, J.C. (2002), On duality theorems for a nondifferentiable minimax fractional programming, *Journal of Computational and Applied Mathematics*, 146, 115–126.
11. Bector, C.R., Chandra, S. and Bector, M.K. (1989), Generalized fractional programming duality: A parametric approach, *Journal of Optimization Theory and Applications*, 60, 243–260.
12. Xu, Z.K. (1988), Saddle-point type optimality criteria for generalized fractional programming, *Journal of Optimization Theory and Applications*, 57, 189–196.